C 40185 **(Pages : 4)** **Name.......................................**

Reg. No...................................

SIXTH SEMESTER (CUCBCSS—UG) DEGREE EXAMINATION MARCH 2023

Mathematics

MAT 6B 09—REAL ANALYSIS

(2017—2018 Admissions)

Time : Three Hours Maximum : 120 Marks

Part A

Answer **all** *questions.*
Each question carries 1 *mark.*

1. Give an example for a continuous function which is not bounded.
2. State Preservation of Intervals Theorem.
3. Define Lipschitz itz function.
4. State Boundedness theorem.
5. Find $\| P \|$ if $P = (0, 1, 1.5, 2, 3.4, 5)$ in a partition of $[0, 5]$.
6. State Lebesgue integrability criterion.
7. Define uniform convergence of a sequence of functions.
8. $\lim_{n \to \infty} \frac{\sin(nx + n)}{n} =$
9. Give an example for an improper integral of second kind.
10. Cauchy principal value of $\int_{-1}^{1} \frac{1}{x}\, dx =$
11. Fill in the blanks : $\Gamma(1) =$ ——————.
12. Define Beta function.

(12 × 1 = 12 marks)

Turn over

Part B

Answer any **ten** *question.*
Each question carries 4 *marks.*

13. Let I be an interval and let $f : \mathrm{I} \to \mathbb{R}$ be continuous on I. If $a, b \in \mathrm{I}$ and if $k \in \mathbb{R}$ satisfies $f(a) < k < f(b)$, then prove that there exists $c \in \mathrm{I}$ between a and b such that $f(c) = k$.

14. Let I be a closed bounded interval and let $f : \mathrm{I} \to \mathbb{R}$ be continuous on I. Then prove that $f(\mathrm{I}) = \{f(x) : x \in \mathrm{I}\}$ is a closed bounded interval.

15. If $f : \mathrm{A} \to \mathbb{R}$ is uniformly continuous on a subset A of $\mathbb{R}$ and if (x_n) is a cauchy sequence in A then prove that $(f(x_n))$ is a cauchy sequence in $\mathbb{R}$.

16. Show that $f(x) = x^2$ on $\mathrm{A} = [0, b], b > 0$ satisfies a Lipschitz condition.

17. Suppose that $f, g \in \Re[a, b]$ and $k \in \mathbb{R}$. Prove that $kf \in \Re\ [a, b]$ and $\int_a^b kf = k\int_a^b f$.

18. Let $\mathrm{G}(x) = \frac{1}{n}$ for $x = \frac{1}{n}\ (n \in \mathbb{N})$ and $\mathrm{G}(x) = 0$ elsewhere on $[0, 1]$. Show that $\mathrm{G} \in \Re[0, 1]$.

19. State and prove composition theorem for Riemann Integrable functions.

20. Let (f_n) be a sequence of bounded functions on $\mathrm{A} \subseteq \mathbb{R}$. Suppose that (f_n) converges uniformly on A to f. Then prove that $\| f_n - f \|_{\mathrm{A}} \to 0$.

21. Discuss the uniform convergence of $\sum_{n=1}^{\infty} \frac{1}{x^2 + n^2}$.

22. State and prove Weierstrass M-Test for a series of functions.

23. Test the convergence of $\int_0^{\infty} \frac{1}{x^2}\, dx$.

24. Show that $\Gamma(n+1) = n!$ when n is a positive integer.

25. Show that $\beta(m, n) = \int_0^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}} dy$.

26. Evaluate $\int_0^1 x^7 (1-x)^8 dx$.

(10 × 4 = 40 marks)

Part C

Answer any **six** *question.*
Each question carries 7 marks.

27. State and Prove Maximum-Minimum Theorem.

28. State and prove Continuous extension theorem.

29. Let $I = [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. If $\in > 0$, then prove that there exists a step function $S_{\in} : I \to \mathbb{R}$ such that $| f(x) - S_{\in}(x) | < \in$ for all $x \in I$.

30. If $f \in \Re[a, b]$ then prove that the value of the integral is uniquely determined.

31. If $f \in \Re[a, b]$, then prove that f is bounded on $[a, b]$.

32. State and prove Cauchy Criterion for Riemann Integrability.

33. State and prove Taylor's Theorem with the Reminder.

34. Prove that $\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \forall m, n > 0$.

35. Show that $\int_0^{\frac{\pi}{2}} \sin^{(p-1)} \theta \cos^{(q-1)} \theta d\theta = \frac{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}{2\Gamma\left(\frac{p+q}{2}\right)}$.

(6 × 7 = 42 marks)

Turn over

Part D

Answer any **two** *question.*
Each question carries 13 *marks.*

36. a) State and prove Location of roots theorem.

    b) Test the uniform continuity of $f(x) = \sqrt{x}$ on $[0, 2]$.

37. a) If f is continuous on $[a, b]$, then Prove that the indefinite integral $F(z) = \int_a^z f$ for $z \in [a, b]$ is differentiable on $[a, b]$ and $F'(x) = f(x)$ for all $x \in [a, b]$.

    b) Show that Dirichlet function is not Riemann Integrable.

38. a) Express the integral $\int_0^{\frac{\pi}{2}} \sin^p x \cos^q x dx$ in terms of Beta function.

    b) Evaluate $\int_0^{\frac{\pi}{2}} \sin^7 x \cos^5 x dx$.

(2 × 13 = 26 marks)