D 50665 **(Pages : 3)** **Name..................................**

Reg. No...............................

FIFTH SEMESTER (CBCSS—UG) DEGREE EXAMINATION NOVEMBER 2023

Mathematics

MTS 5B 05—ABSTRACT ALGEBRA

(2020 Admission onwards)

Time : Two Hours and a Half Maximum : 80 Marks

Section A

Answer any number of questions.
Each question carries 2 marks.
Ceiling is 25.

1. Make multiplication table for $\mathbb{Z}_7$.

2. State and prove Fermat theorem.

3. Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 5 & 4 & 6 & 1 & 7 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 5 & 7 & 4 & 6 & 3 \end{pmatrix}$ be permutation in S_7.

   Find $\sigma\tau$ and $\tau\sigma$.

4. State and prove cancellation property for groups.

5. Is $\mathbb{Z}_8^x$ cyclic ? Justify.

6. Let H be a subgroup of the group G. For $a, b \in$ G, define $a \sim b$ if $ab^{-1} \in$ H. Prove that ~ is an equivalence relation.

7. Find HK in $\mathbb{Z}_{16}^x$, if H = $\langle [3] \rangle$ and K = $\langle [5] \rangle$.

8. Let G_1 and G_2 be groups, and let $\phi : G_1 \to G_2$ be a function such that $\phi\ (ab) = \phi(a)\,\phi(b)$ for all $a, b \in G_1$. Prove that ϕ is one to one if and only if $\phi(x) = e$ implies $x = e$, for all $x \in G_1$.

Turn over

9. Let G be a group, and let $a, b \in$ G be elements such that $ab = ba$. If the orders of a and b are relatively prime, prove that o (ab) = o (a) o (b).

10. Let $\phi : G_1 \to G_2$ be a group homomorphism, with K = ker ϕ. Prove that K is a subgroup of G_1.

11. Let $\phi : G_1 \to G_2$ be an onto homomorphism. If H_1 is normal in G_1, prove that ϕ (H_1) is normal in G_2.

12. Let G = $\mathbb{Z}_{24}$ and H = $\langle [3] \rangle$. Find all cosets of H.

13. State second isomorphism theorem.

14. Prove that Aut $(\mathbb{Z}_n) \cong \mathbb{Z}_n^{\text{x}}$.

15. If D is an integral domain, prove that D $[x]$ is an integral domain.

Section B

Answer any number of questions.

Each question carries 5 *marks.*

Ceiling is 35.

16. Let n be a positive integer. Prove that :

    (a) The congruence class $[a]_n$ has a multiplicative inverse in $\mathbb{Z}_n$ if and only if $(a, n) = 1$.

    (b) A non zero element of $\mathbb{Z}_n$ is either has a multiplicative inverse or is a divisor of zero.

17. (a) Let $\sigma \in S_n$ be written as a product of disjoint cycles, prove that the order of σ is the least common multiple of the lengths of its cycles.

    (b) Find the order of $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 5 & 3 & 2 & 1 \end{pmatrix}$.

18. Let G be a group and let H be a subset of G. Prove that H is a subgroup of G if and only if H is nonempty and $ab^{-1} \in$ H for all $a, b \in$ H.

19. Let G_1 and G_2 be groups. Prove that the direct product $G_1 \times G_2$ is a group under the operation defined for all $(a_1, a_2), (b_1, b_2) \in G_1 \times G_2$ by $(a_1, a_2)(b_1, b_2) = (a_1b_1, a_2b_2)$.

20. If m and n are positive integers such that gcd $(m, n) = 1$, prove that $\mathbb{Z}_{mn}$ is isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_n$.

21. Give the subgroup diagram of $\mathbb{Z}_{12}$.

22. State and prove fundamental homomorphism theorem.

23. Let G be a group. Prove that Aut (G) is a group under composition of functions, and Inn (G) is a normal subgroup of Aut (G).

Section C

Answer any **two** *questions.*
Each question carries 10 *marks.*
Maximum 20 *marks.*

24. If permutation written as a product of transpositions in two ways, prove that the number of transpositions is either even in both cases or odd in both cases.

25. (a) State and prove Lagrange theorem.

    (b) Prove that any group of prime order is cyclic.

26. State and prove Cayley theorem.

27. State and prove second isomorphism theorem.

(2 × 10 = 20 marks)